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A qualitative analysis of the subharmonic oscillations of a parametrically excited flexible rod☆

K.V. Avramov
Kharkov, Ukraine

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Abstract

A non-autonomous non-linear dynamical system with a small parameter that describes the parametric oscillations of a flexible rod with three static equilibrium positions is obtained. The generating equation of this model is a dynamical system in a plane with a separatrix loop. The qualitative analysis presented includes an investigation of the stability and bifurcation of subharmonic motions at resonance energy levels.

This paper focuses on the results of investigations of non-autonomous dynamical systems with a small parameter which are obtained for non-autonomous equations, similar to two-dimensional systems with a separatrix loop. If a perturbation is introduced into an autonomous system with a separatrix loop, the loop can split. The relative distance between the separatrix manifolds that split off is given by Mel'nikov's function.[1] If an intersection of separatrix manifolds is observed in a dissipative system with a small parameter, the steady-state oscillations are chaotic.[2,3] Subharmonic oscillations are investigated to predict the bifurcations that lead to chaos. To analyse them, a non-conservative system, similar to a two-dimensional system with a separatrix loop, is written in action–angle variables.[4,5].

The parametric oscillations of flexible rods were previously examined taking non-linear delay into account in Ref. 6. Forced non-linear oscillations of a flexible rod, which were described by a discrete model with two degrees of freedom, were investigated in Refs. 7,8. Only the saddle-node bifurcations of a flexible rod were considered in 9.

Mel'nikov's asymptotic method is used below for the first time to analyse the subharmonic oscillations of a parametrically excited flexible rod with three static equilibrium positions. The ordinary differential equation obtained for the parametric oscillations of the rod is significantly more complex than a Duffing oscillator or the non-linear Mathieu's equation, which were previously investigated by Mel'nikov's method in Refs. 1–5. The parametric oscillations of a flexible rod are described using a model that takes into account the non-linear relation for the curvature of a curved neutral axis and a non-linear delay. The transition to action–angle variables enabled us to investigate the bifurcation behaviour of the system in detail, which was not done in Ref. 9.

1. Formulation of the problem

Consider a flexible rod with a load of mass M attached to one of its ends (Fig. 1). Rectilinear motion of the load occurs under the action of a periodically varying external force $P_0 + P\cos\bar{\Omega}t$. The displacement of the load during

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E-mail address: kvavr@kharkov.ua.

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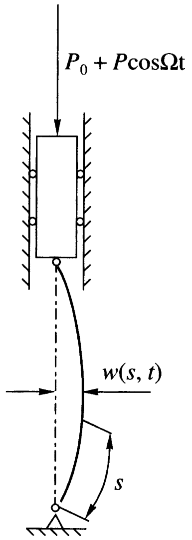

Fig. 1.

transverse oscillations of the rod is denoted by

$$\eta(t) = \frac{1}{2}\int_0^l w'^2 ds$$

where $w = w(s, t)$ is a function that describes the transverse oscillations, and l is the length of the rod. A prime denotes a derivative with respect to the coordinate s along the deformable neutral axis of the rod, and a dot denotes a derivative with respect to the time t. We shall take into account the viscous resistance force $R_L = c_L\eta^\bullet$ applied to the load. The curvature of the neutral line of the rod is related to its transverse displacement in the quadratic approximation by the following expression

$$\frac{1}{\rho} = w'' + \frac{1}{2}w''w'^2$$

The non-linear delay of the rod as it moves is taken into account.

The integrodifferential equation of the transverse oscillations in the cubic approximation with respect to the displacements and their derivatives has the form

$$\begin{gathered} EJw'''' + \frac{EJ}{2}(w''w'^2)'' + \left\{P_0 + P\cos\bar{\Omega}t - \frac{M}{2}\int_0^l (w'^2)^{\cdot\cdot} ds - \frac{c_1}{2}\int_0^l (w'^2)^{\cdot} ds\right\}w'' \\ + cw^{\cdot} + \mu w^{\cdot\cdot} - (Nw')' = 0, \quad N = \frac{\mu}{2}\int_0^{s_1} ds_1 \int_0^{s_1} (w'^2)^{\cdot\cdot} ds_2 \end{gathered} \tag{1.1}$$

Here EJ is the flexural stiffness of the rod, μ is the mass per unit length of the rod, $cw^\bullet$ is the force of resistance to transverse displacements distributed along the rod and N is the longitudinal force in cross-sections of the rod. The left end of the rod is hinged at a fixed point and the right end is pivoted on the load. The boundary conditions therefore have the form

$$w|_{s=0} = w''|_{s=0} = w|_{s=l} = w''|_{s=l} = 0$$

Let us consider the oscillations of the rod when there is a loss of static stability and $P_0 > P_*$, where P_* is the value of the Euler critical force. The rod has three static equilibrium positions.

We introduce the dimensionless parameters and variables

$$\begin{aligned} &\varepsilon\delta = \frac{cl^2}{\sqrt{EJ\mu}}, \quad \varepsilon\delta_1 = \frac{c_1 w_*^2}{2l\sqrt{EJ\mu}}, \quad \varepsilon\Gamma = \frac{l^2 P}{EJ}, \quad \Gamma_0 = \frac{P_0 l^2}{EJ}, \quad \varepsilon\gamma = \frac{w_*^2}{2l^2} \\ &m_1 = \frac{M}{l\mu}, \quad u = \frac{w}{w_*}, \quad \tau = \sqrt{\frac{EJ}{l^4\mu}}t, \quad \xi = \frac{s}{l}, \quad \Omega = \frac{\bar{\Omega} l^2\sqrt{\mu}}{\sqrt{EJ}}; \quad w_* = \frac{2\sqrt{2}}{\pi}l\sqrt{\frac{P_0}{P_*} - 1} \end{aligned} \tag{1.2}$$

where $\varepsilon \ll 1$, and w_* is the magnitude of the static deflection at the centre of the rod, which is introduced to reduce the dynamical problem to dimensionless form.

The dimensionless frequencies of non-linear system (1.1) for $P_0 < P_*$ have the form

$$p_\nu = \nu^2\pi^2, \quad \nu = 1, 2, \ldots$$

The following range of variation of the frequency of the perturbing force is investigated: $\Omega \in [0.5, 4]$. We will represent the oscillations of the rod in the form $u = q(t)\sin \pi\xi$, where $\sin \pi\xi$ is the first mode of the oscillations of a linear system.

Applying Bubnov's method to Eq. (1.1), taking expressions (1.2) into account, we obtain the following ordinary differential equation

$$\begin{aligned} &q^{\cdot\cdot} + \lambda(q^3 - q) + \varepsilon[-\gamma\rho\lambda\pi^4(q^5 - q^3) + \gamma\rho\pi^4 q q^{\cdot 2} + \delta q^{\cdot} - \Gamma\pi^2 q\cos\Omega\tau + \delta_1\pi^4 q^{\cdot} q^2] = 0 \\ &\lambda = \Gamma_0\pi^2 - \pi^4, \quad \rho = m_1 + \frac{1}{3} - \frac{3}{8\pi^2} \end{aligned} \tag{1.3}$$

Since the critical value of the dimensionless longitudinal load Γ_0 is $\Gamma_* = \pi^2$, we have $\lambda > 0$. We emphasize that dynamical system (1.3) provides an approximate description of the parametric oscillations of a flexible rod. The highly simplified system (1.3), which was generated by the problem of the parametric oscillations of a flexible rod, is investigated in this paper.

To simplify the ensuing discussion, we will write system (1.3) in the form

$$x^{\cdot} = f(x) + \varepsilon g(x, t); \quad x = (u, \upsilon) = (q, q^{\cdot}), \quad g = (g_1, g_2)$$

The generating system (1.3) is non-linear and conservative, and in the phase plane, it contains three fixed points: two centres $(q, q^{\bullet}) = (\pm 1, 0)$ and a saddle point $(q, q^{\bullet}) = (0, 0)$, which are shown in Fig. 2. The homoclinic trajectories Γ_1 and Γ_2 are invariant manifolds of the saddle point. Within the homoclinic orbits, the phase plane of the generating system is filled by periodic motions. A value of the elliptic modulus k $(0 < k < 1)$ corresponds to each trajectory. Note that the k values for periodic motions near the homoclinic orbits are close to unity and that $k = 1$ corresponds to a homoclinic trajectory. The period of motion is given by the relation

$$T(k) = 2\mathbf{K}\sqrt{2 - k^2}/\sqrt{\lambda}$$

where $\mathbf{K} = \mathbf{K}(k)$ is a complete elliptic integral of the first kind. Note that $\lim T(k) = \infty$ as $k \to 1$.

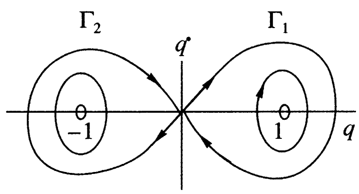

Fig. 2.

2. A qualitative investigation of the motions

Let us consider the subharmonic oscillations in system (1.3) that satisfy the resonance condition

$$T(k) = mT_1 \tag{2.1}$$

where $T_1 = 2\pi/\Omega$ is the period of the parametric action. Following the approach adopted previously,[5] we will use the canonical transformations $I = I(u, v)$ and $\theta = \theta(u, v)$, which are associated with the transition to the action–angle variables I and θ. Then, in I, θ variables, dynamical system (1.3) takes the form

$$\begin{gathered} I^{\bullet} = \varepsilon F(I, \theta, t), \quad \theta^{\bullet} = \omega(I) + \varepsilon G(I, \theta, t) \\ F(I, \theta, t) = \frac{\partial I}{\partial u}g_1 + \frac{\partial I}{\partial v}g_2, \quad G(I, \theta, t) = \frac{\partial \theta}{\partial u}g_1 + \frac{\partial \theta}{\partial v}g_2, \quad \omega(I) = \frac{\partial H}{\partial I} \end{gathered} \tag{2.2}$$

where H is the Hamiltonian of the generating system.

Note that the solutions of generating system (2.2) are periodic ($I = I_0 = \text{const}$, $\theta = \omega(I_0)t + \theta_0$). They are shown in Fig. 2 within the homoclinic orbits Γ_1 and Γ_2.

For the integers $m = 1, 2, \ldots$ we determine the values of I^m from resonance conditions (2.1). Then we examine the motions near the resonance energy levels.[5] We define these oscillations by the relations

$$I = I^m + \sqrt{\varepsilon}h(t), \quad \theta = \omega(I^m)t + \phi \tag{2.3}$$

We substitute expressions (2.3) into system (2.2). We treat the system of equations found by applying the method of averaging[10] using the replacement of variables

$$(h, \phi) = (\bar{h}, \bar{\phi}) + \sqrt{\varepsilon}\Phi(\bar{h}, \bar{\phi}, t)$$

where $(\bar{h}, \bar{\phi})$ are new variables. As a result, we obtain the dynamical system

$$\bar{h}^{\bullet} = \frac{\sqrt{\varepsilon}}{2\pi}\bar{M}\left(\frac{\bar{\phi}}{\omega}\right) + \varepsilon\frac{\partial \bar{F}}{\partial I}\bar{h}, \quad \bar{\phi}^{\bullet} = \sqrt{\varepsilon}\frac{\partial \omega}{\partial I}\bar{h} + \varepsilon\left[\frac{1}{2}\frac{\partial^2 \omega}{\partial I^2}\bar{h}^2 + \bar{G}(\bar{\phi})\right] \tag{2.4}$$

where

$$\begin{gathered} \bar{M}(t_0) = -\delta\sqrt{\lambda}J_1(k) + \Gamma\pi^2 J_3(k)\sin\Omega t_0 - \delta_1\pi^4\sqrt{\lambda}J_2(k) \\ J_1(k) = \frac{1}{\sqrt{\lambda}}\int_0^{mT} q_0^{\bullet 2}\,dt = \frac{4}{3}[(2-k^2)\mathbf{E} - 2k'^2\mathbf{K}](2-k^2)^{-3/2} \\ J_2(k) = \frac{1}{\sqrt{\lambda}}\int_0^{mT} q_0^{\bullet 2} q_0^2\,dt = \frac{8}{15}[2(k^4 + k'^2)\mathbf{E} + (k^2 - 2)k'^2\mathbf{K}](2-k^2)^{-5/2} \\ J_3\sin\Omega t_0 = \int_0^{mT} q_0 q_0^{\bullet}\cos(\Omega\tau + \Omega t_0)\,d\tau = \frac{\Omega^2\pi}{\lambda}\operatorname{csch}\frac{m\pi\mathbf{K}'}{\mathbf{K}}\sin\Omega t_0 \\ \omega = \omega(I^m), \quad \frac{\partial \omega}{\partial I} = -\frac{\sqrt{\lambda}\pi^2(2-k^2)[(2-k^2)\mathbf{E} - 2k'^2\mathbf{K}]}{2k^4k'^2\mathbf{K}^3} \\ \frac{\partial^2 \omega}{\partial I^2} = -\frac{\sqrt{\lambda}\pi^3(2-k^2)^{5/2}}{4k^8k'^4}\left[2(k'^6 + 3k'^2 + k^4)\frac{\mathbf{E}}{\mathbf{K}^4} + k'^2(4k'^2 - k^4)\frac{1}{\mathbf{K}^3} - 3(2-k^2)^2\frac{\mathbf{E}^2}{\mathbf{K}^5}\right] \end{gathered} \tag{2.5}$$

The bar over F and G denotes averaging of these functions, $k'^2 = 1 - k^2$, $\mathbf{K}' = \mathbf{K}(k')$, $(q_0, q_0^{\bullet})$ denotes the solutions of generating system (1.3), and $\mathbf{E} = \mathbf{E}(k)$ is a complete elliptic integral of the second kind. System (2.4) was obtained using the relation

$$mT\omega(k)\frac{\partial \bar{F}}{\partial I} = \bar{M}\left(\frac{\bar{\phi}}{\omega}\right)$$

To continue the analysis, we rewrite system (2.4) in the form

$$\begin{aligned}\dot{\bar{h}} &= \frac{\sqrt{\varepsilon}}{2\pi}(-(\delta_1-\delta_*)\pi^4\sqrt{\lambda}J_2+\Gamma\pi^2 J_3\sin m\bar{\phi})+\varepsilon[\chi+\Gamma\pi^2 K_1\sin m\bar{\phi}]\bar{h}\\ \dot{\bar{\phi}} &= \sqrt{\varepsilon}\frac{\partial\omega}{\partial I}\bar{h}+\varepsilon\left[\frac{\partial^2\omega}{2\partial I^2}\bar{h}^2-\frac{\Gamma\pi^2 K_1}{m}\cos m\bar{\phi}\right]\end{aligned} \tag{2.6}$$

where

$$\delta_* = -\frac{\delta J_1(k)}{\pi^4 J_2(k)},\quad \chi = \delta\sqrt{\lambda}\frac{\Omega(2-k^2)^2\sigma(k)}{60\pi m\lambda k^3 J_2}-\pi^4(\delta_1-\delta_*)\sqrt{\lambda}K_2$$

$$K_1 = \frac{\Omega\pi}{\lambda}\operatorname{csch}\frac{m\pi\mathbf{K}'}{\mathbf{K}}\left[\frac{(2-k^2)^3\Omega^2\pi}{8\lambda k^4 k'^2 K^2}\operatorname{cth}\frac{m\pi\mathbf{K}'}{\mathbf{K}}+\gamma_1(k)\right],\quad \gamma_1(k) = -\frac{m}{2\pi}\frac{\partial\omega}{\partial I}$$

$$K_2 = \frac{2E\Omega}{\lambda\pi m\sqrt{2-k^2}}+\frac{\omega(k)}{\Omega}J_2$$

$$\sigma(k) = 80(2-k^2)\mathbf{E}^2(k)-160k'^2\mathbf{K}(k)\mathbf{E}(k)-32(k^4+k'^2)\mathbf{K}(k)\mathbf{E}(k)+16k'^2(2-k^2)\mathbf{K}^2(k)$$

The trajectories of dynamical system (2.6) in a plane correspond to the motions of non-autonomous dynamical system (2.2) near the resonance energy levels, as follows from relations (2.3). Therefore, from the properties of the fixed points and the trajectories of system (2.6) we can draw a conclusion about the properties of the periodic orbits and other motions of system (2.2).

We will use the results of the well-developed theory of dynamical systems in a plane[3,11] to analyse system (2.6). We will represent the fixed points of this system in the form

$$(\bar{\phi}_\nu, \bar{h}_\nu) = \left(\frac{(-1)^\nu}{m}\arcsin a+\frac{\pi\nu}{m}, 0\right)+O(\sqrt{\varepsilon}),\quad \nu\in Z,\quad a = \frac{\sqrt{\lambda}}{\Gamma\pi^2 J_3}(J_1\delta+\delta_1\pi^4 J_2) \tag{2.7}$$

Let us investigate the stability of the fixed points. For this purpose we linearize system (2.6) near the fixed points and determine the characteristic exponents. We divide the equilibrium states into two groups according to their characteristic exponents. The saddle fixed points in the phase plane are denoted by α. They have the following characteristic exponents

$$\bar{\sigma}_{1,2} = \pm R+O(\varepsilon);\quad R = \left[\frac{\varepsilon}{2}\left|\frac{\partial\omega}{\partial I}\right|\Gamma\pi J_3 m\sqrt{1-a^2}\right]^{1/2} \tag{2.8}$$

The stability of the fixed points β is specified by the following characteristic exponents

$$\lambda_{1,2} = \frac{1}{2}\operatorname{tr}\tilde{A}\pm iR \tag{2.9}$$

where $\operatorname{tr}\tilde{A}$ is the trace of the matrix of the linearized system that satisfies the relation

$$\lim_{k\to 1}\operatorname{tr}\tilde{A} = \lim_{k\to 1}\varepsilon\frac{\sqrt{\lambda}(\delta_1\pi^4 J_2+J_1\delta)}{2mT_1 k'^2 K(k)} \tag{2.10}$$

Since the values of the elliptic modulus k for motions near resonance energy levels are close to unity, we can carry out an approximate analysis of $\lambda_{1,2}$ using relation (2.10). Then the fixed point β is stable for $\delta_1 < \delta_*(m)$ and is unstable for $\delta_1 > \delta_*(m)$.

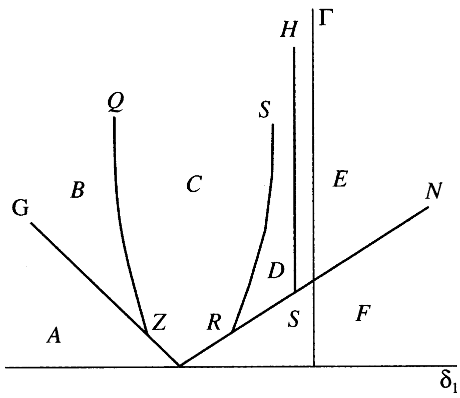

Fig. 3.

Let us examine the bifurcations of system (2.6). Fig. 3 shows a qualitative representation of the bifurcation set in the (δ_1, Γ) parameter plane. The motions of the system in regions *A* and *B* are qualitatively different, because these regions are separated by a saddle-node bifurcation line (*GZ*). A qualitative representation of the phase trajectories is given in Fig. 4b. This figure contains a saddle fixed point (point α), a stable fixed point (β) and a stable periodic motion (L_1). As a result of the saddle-node bifurcation *GZ*, these fixed points merge and then vanish. Therefore, there are no fixed points in region *A* (Fig. 3), and only the stable periodic motions L_1 are observed here (Fig. 4a). A similar bifurcation pattern is observed on passing from region *E* to region *F*. These regions are separated by a saddle-node bifurcation line *RN*. The phase trajectories in regions *E* and *F* are not shown.

We will investigate the heteroclinic orbits of system (2.6). For this purpose, we select the parameter δ_1 in the following way

$$\delta_1 = \delta_*(m) + \sqrt{\varepsilon}\Delta; \quad \Delta = O(1) \tag{2.11}$$

We substitute expression (2.11) into system (2.6). Then the Hamiltonian of the generating system becomes

$$H = \frac{\sqrt{\varepsilon}}{2}\frac{\partial\omega}{\partial I}\bar{h}^2 + \frac{\sqrt{\varepsilon}\Gamma\pi J_3}{2m}\cos m\bar{\phi} \tag{2.12}$$

Dynamical system (2.12) has the following fixed points: the centres $(\bar{\phi}_\nu, \bar{h}_\nu) = \left(\frac{2\nu}{m}\pi, 0\right)$ $(\nu = 0, \pm 1, \ldots)$ and the saddle points $(\bar{\phi}_\nu, \bar{h}_\nu) = \left(\frac{2\nu+1}{m}\pi, 0\right)$. We note that the saddle points are joined to one another by heteroclinic orbits.

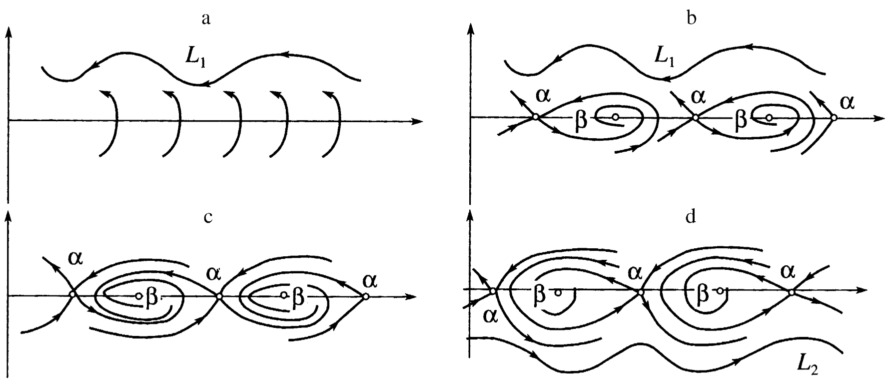

Fig. 4.

Taking into account expression (2.11), we determine the heteroclinic motions in dissipative system (2.6) using Mel'nikov's function[3]

$$\bar{M} = -\frac{\sqrt{\varepsilon}}{2}\frac{\partial\omega}{\partial I}\Delta\pi^3\sqrt{\lambda}J_2\int_{-\infty}^{\infty}\bar{h}dt + \sqrt{\varepsilon}\frac{\partial\omega}{\partial I}\chi|_{\Delta=0}\int_{-\infty}^{\infty}\bar{h}^2dt \tag{2.13}$$

The integration in (2.13) is performed along the heteroclinic orbits of system (2.12). Heteroclinic bifurcations are observed in system (2.6) if

$$\Delta = \pm\frac{4\chi|_{\Delta=0}}{J_2\sqrt{\lambda}}\sqrt{\frac{2\Gamma J_3}{m\pi^7\left|\frac{\partial\omega}{\partial I}\right|}} \tag{2.14}$$

Fig. 3 presents a qualitative representation of the heteroclinic bifurcation sets ZQ and RS. Let us examine the bifurcations observed when the system passes from region B into regions C and D. Region B contains the periodic motion L_1 (Fig. 4b), which "converges" onto a heteroclinic orbit and vanishes. Note that such a heteroclinic trajectory is observed on the bifurcation line ZQ. There are no periodic motions in region C (Fig. 4c). A heteroclinic orbit is observed on the bifurcation line RS, and a periodic motion L_2 is generated in region D (Fig. 4d).

System (2.6) with homoclinic trajectories was obtained using the method of averaging. This method was applied to the system of non-autonomous equations that was derived from system (2.2) using the replacement of variables (2.3). For some parameter values, the homoclinic trajectories in autonomous system (2.6) correspond to intersections of separatrix manifolds of saddle periodic motions of the non-autonomous equations. As a consequence of these intersections, Smale horseshoes appear in the phase space. This phenomenon was investigated in a Duffing–van der Pol oscillator in Ref. 12. The intersection of separatrix manifolds in system (1.3) is beyond the scope of this paper.

Let us consider the change in the stability of the fixed point β that is observed on the bifurcation line HS (Fig. 3). To analyse the trajectories in the vicinity of this fixed point, we introduce the replacement of variables $(\psi, h) = (\phi - a, h)$. Then the fixed point has the coordinates $(\psi, h) = (0, 0)$. Examining the motion near the equilibrium state ($|\psi| \ll 1$), we obtain the differential equation

$$\psi^{\cdot\cdot} + \omega_1^2\psi = \sqrt{\varepsilon}\psi^{\cdot}(\alpha + r\psi) + O(\varepsilon) \tag{2.15}$$

where

$$\omega_1^2 = \frac{1}{2}\left|\frac{\partial\omega}{\partial I}\right|\Gamma\pi J_3 m\sqrt{1-(\delta_1-\delta_*)^2\beta_1^2}$$

$$\alpha = \chi + 2\Gamma\pi^2K_1(\delta_1-\delta_*)\beta_1, \quad \beta_1 = \frac{\pi^2\sqrt{\lambda}J_2}{\Gamma J_3}$$

$$r = \left[\frac{1}{2}\left(\frac{\partial\omega}{\partial I}\right)^{-1}\frac{\partial^2\omega}{\partial I^2}\Gamma\pi J_3 m + 2\Gamma\pi^2K_1m\right]\sqrt{1-(\delta_1-\delta_*)^2\beta_1^2}$$

We solve Eq. (2.15) by the Krylov–Bogolyubov method[10] using the replacement of variables

$$(\psi, \psi^{\cdot}) = (a\cos\theta, -a\omega\sin\theta)$$

We then obtain the following system of equations in a and θ

$$a^{\cdot} = \sqrt{\varepsilon}\alpha\frac{a}{2} + O(\varepsilon), \quad \theta^{\cdot} = \omega_1 + O(\varepsilon) \tag{2.16}$$

where

$$\alpha(k) = \delta\sqrt{\lambda}\frac{\Omega(2-k^2)^2\sigma(k)}{60\pi m\lambda k^3J_2} + (\delta_1-\delta_*)\pi^4\sqrt{\lambda}\left(2K_1\frac{J_2}{J_3} - K_2\right)$$

Hence it follows that the fixed point $(\psi, h) = (0, 0)$ is asymptotically stable for $\alpha < 0$ and is unstable for $\alpha > 0$. The bifurcation line that separates the asymptotically stable fixed points from the unstable fixed points is $\alpha = 0$. The equation of the bifurcation line can be written in the form

$$\delta_1 - \delta_* = \Delta_*(k)$$

$$\lim_{k \to 1} \Delta_*(k) = \frac{15\delta}{8\pi^4\sqrt{\lambda}}\left[\frac{\Omega\pi}{\lambda}\operatorname{cth}\frac{\Omega\pi}{2\sqrt{\lambda}} + 1\right]^{-1}\lim_{k \to 1} k'^2\mathbf{K}^2 \tag{2.17}$$

It follows from system (2.16) that despite the change in the stability along the bifurcation line *HS*, limit cycles are not generated.

Parametric oscillations of a flexible rod give rise to saddle-node bifurcations of the subharmonic regimes. The bifurcations of the system are qualitatively identical for subharmonic oscillations (2.1) of any order *m*. It follows from our earlier paper[9] that the oscillations of a flexible rod are chaotic after these bifurcations. These motions are accompanied by snapping of the rod between three static equilibrium positions.

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